6. Data Structures

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https://fm-dcc.github.io/alg2425

U.PORTO
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CISTER - Research Centre in
Real-Time & Embedded
Computing Systems

Where we are

- Algorithm Correctness
- Complexity: worst/best-case analysis
- Asymptotic analysis

- Recursive algorithms
- Average-case and randomized algorithms
- Dynamic programming
- Amortized analysis
- Data structures

Overview

- Sets and Sequences
- Buffers:
  - Stacks
  - Queues
  - Priority queues
- Dictionaries
  - Hashtables
  - Search trees

Motivation

We have seen that

Different data structures are better at different operations

We will see

Useful data structures and associated operations (code)

Examples

Arrays can have operations to implement sets, multisets, trees, etc.

Sets and Sequences

Sets and Multisets

```
#define MAXS 100
typedef int SetInt [MAXS] ;
```

Given `SetInt s`:

$5 \in s \Leftrightarrow$ `s[5]!=0`

```
#define MAXMS 100
typedef int MSetInt [MAXS] ;
```

Given `MSetInt ms`:

$\{4, 4\} \subseteq$ `ms` $\Leftrightarrow$ `ms[4]` ≥ 2

```
void initSet        (SetInt);
int  searchSet      (SetInt, int);
int  addSet         (SetInt, int);
int  emptySet       (SetInt);
void unionSet       (SetInt, SetInt,
                     SetInt);
void intersectSet   (SetInt, SetInt,
                     SetInt);
void differenceSet(SetInt, SetInt,
                     SetInt);
```

```
void initMSet        (MSetInt);
int  searchMSet      (MSetInt, int);
int  addMSet         (MSetInt, int);
int  emptyMSet       (MSetInt);
void unionMSet       (MSetInt, MSetInt,
                      SetInt);
void intersectMSet   (MSetInt, MSetInt,
                      MSetInt);
void differenceMSet(MSetInt, MSetInt,
                      MSetInt);
```

Ex. 6.1: What is the expected cost of each function? Could you implement them?

```c
typedef struct list {
  int value ;
  struct list *next;
} *LInt;
```

```c
LInt add (int x, LInt l) {
  LInt new =
    malloc(sizeof(struct list));
  if (new != NULL) {
    new->value=x;
    new->next=l ;
  }
  return new;
}
```

```c
LInt dda (int x, LInt l) {
  LInt pt = l;
  while (pt != NULL) pt = pt->next;
  pt = malloc(sizeof(struct list));
  pt -> value = x;
  pt -> next = NULL ;
  return l ;
}
```

Sequences – Recall linked lists (fixed)

```
typedef struct list {
  int value ;
  struct list *next;
} *LInt;
```

```
LInt add (int x, LInt l) {
  LInt new =
    malloc(sizeof(struct list));
  if (new != NULL) {
    new->value=x;
    new->next=l ;
  }
return new;
}
```

```
LInt dda (int x, LInt l) {
  LInt pt = l, prev;
  while (pt != NULL) {
    prev = pt; pt = pt->next; }
  pt = malloc(sizeof(struct list));
  pt->value = x;
  pt->next = NULL ;
  if (l==NULL)  l = pt;
  else prev->next = pt;
  return l;
}
```

Ex. 6.2: What is the possible complexity of `lookup`, `concat`, `reverse`?

```c
typedef struct list {
  int value ;
  struct list *next;
} *LInt;
```

Idea for `reverseRec`

1. reverse the tail
2. add head to the end

```
reverseRec([])           = []
reverseRec([x1,x2,...]) = dda(x1,reverseRec([x2,...]))
```

```
LInt reverseRec (LInt l) {
  LInt r, pt;
  if (l==NULL || l->next==NULL)
      r=l;
  else {
    r = pt = reverseRec (l->next);
    while (pt->next != NULL)
      pt = pt->next;
    pt->next = l;
    l->next = NULL;
  }
  return r; }
```

```
LInt reverseLoop (LInt l) {
  LInt r, tmp;
  r = NULL;
  while (l != NULL) {
    tmp=l; l=l->next;
    tmp->next=r; r=tmp;
  }
  return r;
}
```

Ex. 6.3: What is the complexity of each `reverse`?

Ex. 6.4: What is the (informal) loop invariant in `reverseLoop`, assuming:
`pre:l==l`$_0$ and `post:r==rev(l`$_0$`)`?

`https://docs.scala-lang.org/overviews/collections-2.13/performance-characteristics.html`

Buffers (stacks and queues)

Stacks

```c
#define MAX 1000
typedef struct stack {
  int values [MAX];
  int sp;
} Stack;
```

```c
typedef struct cell {
  int value;
  struct cell *next;
} Cell , *Stack;
```

```c
typedef struct stack {
  int size;
  int *values;
  int sp;
} Stack;
```

Stacks

```c
#define MAX 1000
typedef struct stack {
  int values [MAX];
  int sp;
} Stack;
```

with static arrays

```c
typedef struct cell {
  int value;
  struct cell *next;
} Cell , *Stack;
```

with linked lists

```c
typedef struct stack {
  int size;
  int *values;
  int sp;
} Stack;
```

with dynamic arrays

Ex. 6.5: (Informally) what is the expected complexity of: `push`, `pop`, `head`?

```c
void push (Stack *s , int x){
  if (s->sp == s->size)
    doubleArray (s);
  s->values[s->sp++] = x;
}

void doubleArray (Stack *s){
  s->size *= 2;
  s->values =
    realloc(s->values, s->size);
}
```

```c
int pop (Stack *s){
  // reduces by half when only
  // 25% capacity is used
  ...
}

void halfArray (Stack *s){
  ...
}
```

Ex. 6.6: Implement the optimised pop function and discuss its complexity.

Queues

```c
#define MAX 1000
typedef struct queue
{
  int values [MAX];
  int start , size;
} Queue;
```

```c
typedef struct cell {
  int value ;
  struct cell *prox ;
} Cell ;

typedef struct queue {
  struct cell *start , *end;
} Queue;
```

```c
typedef struct queue
{
  int max;
  int *values;
  int start , size;
} Queue;
```

Queues

```c
#define MAX 1000
typedef struct queue
{
  int values [MAX];
  int start, size;
} Queue;
```

with static arrays (circular)

```c
typedef struct cell {
  int value ;
  struct cell *prox ;
} Cell ;

typedef struct queue {
  struct cell *start, *end;
} Queue;
```

with linked lists

```c
typedef struct queue
{
  int max;
  int *values;
  int start, size;
} Queue;
```

with dynamic arrays (circular)

Ex. 6.7: (Informally) what is the complexity of: `init`, `isEmpty`, `enqueue`, `dequeue`?

- Binary tree
- Each node is smaller than any of its children
- Implemented as an array

```
#define MAX 1000
typedef struct prQueue {
  int values [MAX];
  int size ;
} PriorityQ ;
```

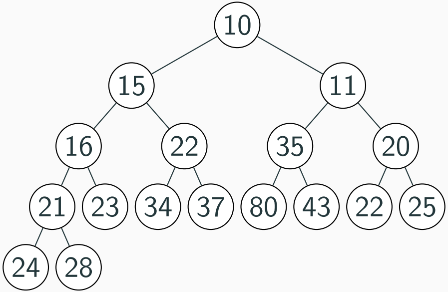
Tree example in the board

size=17	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
values:	[10	15	11	16	22	35	20	21	23	34	37	80	43	22	25	24	28]

Priority queue – example

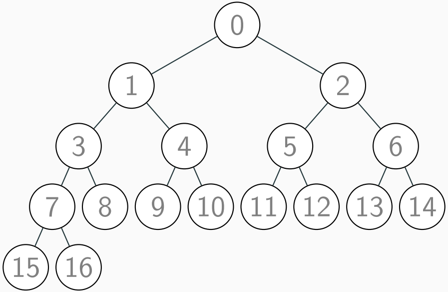
Tree example

```
size=17    0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16
values:  [10 15 11 16 22 35 20 21 23 34 37 80 43 22 25 24 28]
```



Tree example

```
   size=17   0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16
  values:  [10 15 11 16 22 35 20 21 23 34 37 80 43 22 25 24 28]
```



Ex. 6.8: Using the previous example, provide an expression to:

1. calculate the index of the *left* tree given a position `i`
2. calculate the index of the *right* tree given a position `i`
3. calculate the index of the *parent* of a given a position `i`
4. calculate the index of the *first leaf*

Ex. 6.9: Define `bubbleUp(int i, int h[])`

Used to add elements. Fixes a min-heap by swapping the `i`-th element with the parent while needed.

Ex. 6.10: Define `bubbleDown(int i, int h[], int N)`

Used to remove elements. Fixes a min-heap by swapping the `i`-th element with one of the children while needed.

Ex. 6.11: Define the following operations:

- `void empty (PriorityQueue *q)` – initialises the queue;
- `int isEmpty (PriorityQueue *q)` – tests if q is empty;
- `int add (int x, PriorityQueue *q)` – adds a value `x`, returning 0 when the queue is full;
- `int remove (PriorityQueue *q, int *rem)` – removes the next element, and copies it to *rem*.

Dictionaries

Hashtables (hashsets)

Dictionary: maps keys to values (hashset: just the keys)
(Keys are unique)

Idea - using a hash function

- *Magic* function `hash` converts a key into an index (number).
- This index points to the position of an array where the value *should* be found.
- Usually the size of the array is less than the set of possible keys, i.e., `hash` is not injective.
- If 2 keys have the same `hash` value, there is a colision that must be mitigated (alternative solutions exist).

Hashtables: Closed and Open Addressing (also hashsets))

Closed Addressing (or chaining)

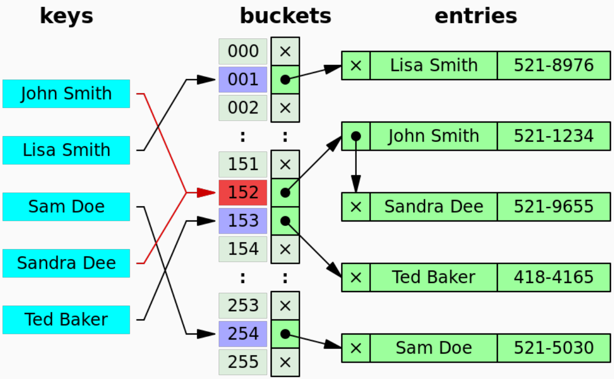
- Table = *array of linked lists*
- Find value of key `k`:
  - go to index `hash(k)`
  - traverse list until `k`

Open Addressing

- Table = *just an array*
- Find value of key `k`:
  - go to index `hash(k)`
  - "*jump*" until `k`

Some concerns

- Use dynamic arrays (grow when the load factor (#keys/HSIZE) gets high)
  - Need to *rehash*
- Smart *jumps* (`probe` function to know where to jump)
- Need to *garbage collect* in open addressing



(from Wikipedia)

Hashtables with Closed Addressing

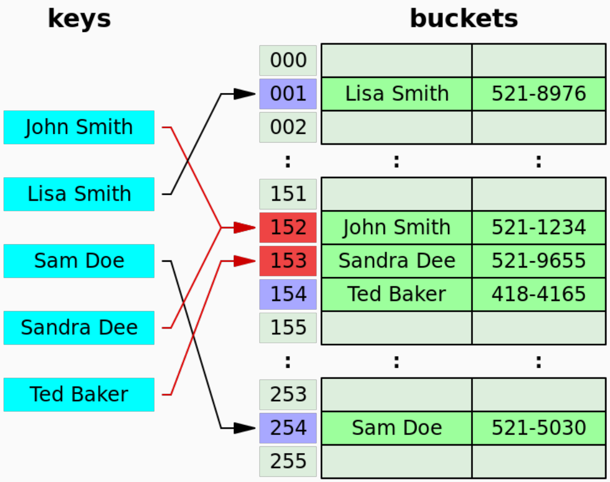
- `int hash(int k, int size);`
- `void initTab(HTChain h);`
- `int lookup(HTChain h, int k, int *i);`
- `int update(HTChain h, int k, int i);`
- `int remove(HTChain h, int k);`

```
#define HSIZE 1000

typedef struct bucket {
  int key;
  int info;
  struct bucket *next;
} *Bucket;

typedef Bucket
   HTChain[HSIZE];
```

Ex. 6.12: Implement `hash` and `lookup`

Ex. 6.13: (Informally) what is the expected complexity of each function?

Intuition: Hashtables with Open Addressing



(from Wikipedia)

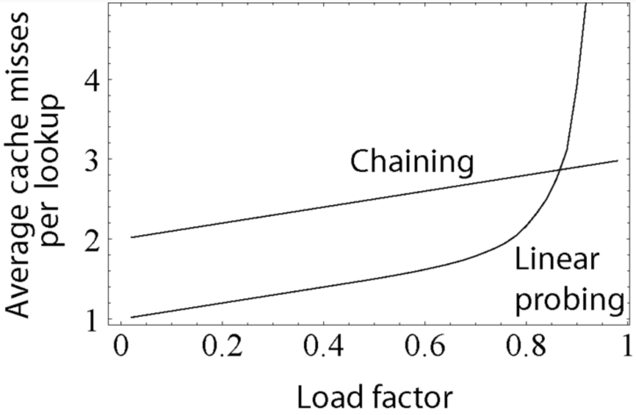
Hashtables with Open Addressing

- `int hash(int k, int size);`
- `void initTab(HashTable h);`
- `void lookup(HashTable h, int k, int *i);`
- `void update(HashTable h, int k, int i);`
- `void remove(HashTable h, int k);`
- `int find_probe (HashTable h, int k)`
  - linear vs. quadratic probing (why quadratic?)

```c
#define HSIZE 1000
#define STATUSFREE 0
#define STATUSUSED 1

typedef struct bucket {
  int status ;
  int key;
  int info;
} Bucket ;

typedef Bucket
  HashTable [HSIZE];
```

Ex. 6.14: Define a linear probing function and update.

Lookups: Open vs. Closed



(from Wikipedia)

Removing with Open Addressing

- `int hash(int k, int size);`
- `void initTab(HashTable h);`
- `void lookup(HashTable h, int k, int *i);`
- `void update(HashTable h, int k, int i);`
- `int find_probe (HashTable h, int k);`
- `void remove(HashTable h, int k);`

```c
#define HSIZE 1000
#define STATUSFREE 0
#define STATUSUSED 1
#define STATUSDEL 2

typedef struct bucket {
  int status ;
  int key;
  int info;
} Bucket ;

typedef Bucket
  HashTable [HSIZE];
```

Ex. 6.15: How would you implement update?
How would you implement a *garbageCollect* that removes deleted cells?
What is their complexity?

Dictionaries with trees – not for evaluation

More dictionaries: balanced trees

We will see:

- Height- and weight-balanced tree
- Self-balancing binary search tree
  - AVL tree
  - Red-black tree

Height-balanced

- more used
- AVL: left-height $=$ right-height $\pm$ 1
- Red-black: similar wrt *black*
- height $=$ log n

Weight-balanced

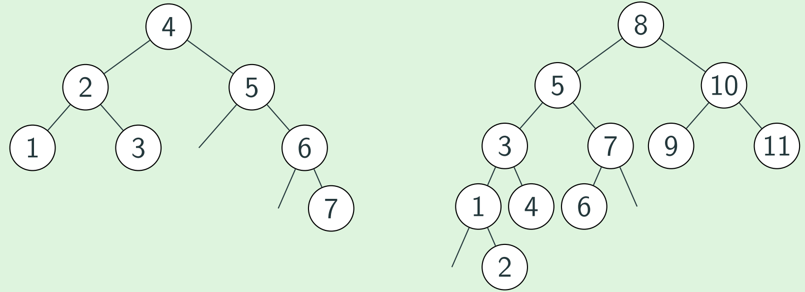
- less used
- leafs-left/right $\geq \alpha \times$ leafs, $0 < \alpha < 1$
- better for lookup intensive systems

AVL trees

- By Adelson-Velsky and Landis
- Oldest self-balancing binary search tree data structure to be invented ('62)
- Binary (left-right) search (sorted) tree
- Labels in the nodes
- At every node, the height of left and right trees differ at most by 1
- Insertions and removals preserve this

Function	Amortized	Worst Case	*Amortized (RB)*	*Worst case (RB)*
Search	$\Theta(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$
Insert	$\Theta(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$	$\mathcal{O}(\log n)$
Delete	$\Theta(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$	$\mathcal{O}(\log n)$

Ex. 6.16: Are these balanced?



See animation

https://en.wikipedia.org/wiki/AVL_tree#/media/File:AVL_Tree_Example.gif

4 rotations: left, right, right-left, right-right

```c
typedef struct avl {
  int bal;
  int key, info;
  struct avl *left , *right ;
} *AVL;
#define LEFT  -1
#define RIGHT 1
#define BAL   0

// returns 0 if key already existed
int updateAVL (AVL *a, int k, int i);
```

Ex. 6.17: How would you implement an **update without balancing**?

Ex. 6.18: How would you implement `AVL rotateRight(AVL a)`?

Full code: updateAVL

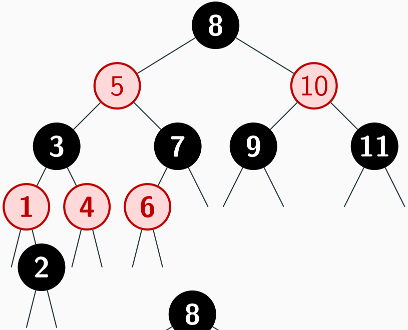
```
int updateAVL(AVL *a,
                int k, int i){
  int g, u;
  *a =
    updateAVLRec(*a,k,i,&g,&u);
  return u;
}
```

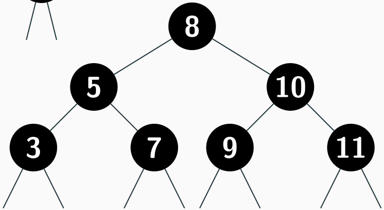
```
AVL updateAVLRec(AVL a , int k ,
                  int i, int *g , int *u){
 if (a == NULL) {          // insert k->i here
   a = malloc (sizeof (struct avl ));
   a->key=k; a->info=i ; a->bal=BAL;
   a->left=a->right=NULL; *g=1; *u=0;
 } else if (a->key==k) {       // update k->i
   a->info=i; *g=0; *u=1;
 } else if (a->key > k) {      // update left
   a->left = updateAVLRec(a->left,k,i,g,u);
   if (*g == 1) switch (a->bal){ // balance
     case LEFT: a= fixLeft(a); *g=0; break;
     case RIGHT:a->bal=BAL;    *g=0; break;
     case BAL:  a->bal=LEFT;         break;
   }
 } else{ // a->key < k          update right
   // left <--> right
 }
 return a ;
}
```

Full code: updateAVL – fix-left

```
AVL fixLeft(AVL a) {
  AVL b, c;
  b=a->left ;
  if (b->bal==LEFT){
    a->bal = b->bal = BAL;
    a=rotateRight(a);
  } else {
    c = b->right ;
    switch (c->bal) {
      case LEFT:  a->bal=RIGHT; b->bal=BAL;  break;
      case RIGHT: a->bal=BAL;   b->bal=LEFT; break;
      case BAL:   a->bal=BAL;   b->bal=BAL;
    }
    c->bal = BAL;
    a->left = rotateLeft(b);
    a = rotateRight(a);
  }
  return a;
}
```

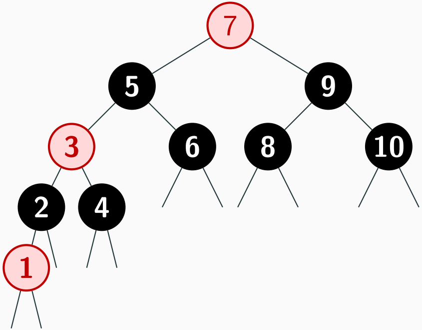
Red-Black Trees





- **1.** Nodes are black or red
- **2.** Empty nodes count as black
- **3.** Red nodes have only black children
- **4.** All down paths from a root have equal black-height

- The root is black.
- Only 1 on the left is a RB tree



- 6 cases for insertion (with nesting)
- 6 cases for deletion (with nesting)

Properties

- height is $\mathcal{O}(\log n)$.
- no path from the root to a leaf is more than twice as long as a path to another leaf

Function	Amortized (AVL)	Worst Case (AVL)	***Amortized***	***Worst case***
Search	$\Theta(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$
Insert	$\Theta(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$	$\mathcal{O}(\log n)$
Delete	$\Theta(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$	$\mathcal{O}(\log n)$

Red-Black trees height is in $\mathcal{O}(\log n)$

- **1.** Nodes are black or red
- **2.** Empty nodes count as black
- **3.** Red nodes have only black children
- **4.** All down paths from a root have equal black-height

Lemma: size of a subtree – $size(x) \geq 2^{bh(x)} - 1$

- $bh(x)$ is the black-height of a node x
- base case: $2^{bh(\text{NULL})} - 1 = 2^0 - 1 = size(\text{NULL})$
- inductive case: For each child c of x:

$$bh(c) = bh(x) \text{ or } bh(c) = bh(x) - 1.$$

Then $size(x) \geq \; 2 \times (2^{bh(x)-1} - 1) + 1$
$= \; 2^{bh(x)-1+1} - 2 + 1 \; = \; 2^{bh(x)-1}$

Theorem: Height of a RB tree is $\mathcal{O}(\log n)$

- Let h be the height of a RB tree x
- For any trace $x, \ldots, leaf$, more than half are black
- $\Rightarrow \; bh(h) \geq h/2$
- $\Rightarrow \; size(x) \geq 2^{h/2} - 1 \; \Leftrightarrow \; h \leq 2 \log(n+1) \in \mathcal{O}(\log n)$